Traveling wave solutions to the free boundary incompressible Navier-Stokes equations

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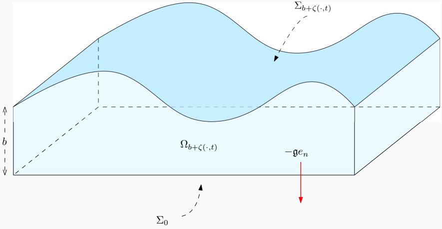
Introduction

Model single layer traveling waves problem



- Single layer of viscous incompressible fluid in a horizontally infinite strip-like domain, in dimension $n \geqslant 2$.
- Bounded below by a flat rigid surface and above by a moving free surface, subject to the effects of gravity and surface tension.
- Fluid is acted upon by a bulk force and a surface stress that are both stationary in a coordinate system moving parallel at speed γ to the fluid bottom.
- Physical example: water waves travel through a canal acted upon by wind.

Fluid domain: setup

- Horizontal cross-section $\Sigma \subseteq \mathbb{R}^{n-1}$
- Equilibrium depth parameter $b > 0$
- Equilibrium domain $\Omega_b = \Sigma \times (0, b)$
- Unknown time-dependent free surface function $\zeta : \Sigma \times [0, \infty) \to (-b, \infty)$
- Time-dependent upper free surface: $\Sigma_{b+\zeta(\cdot,t)} = \{x_n = \zeta(x', t) \text{ for some } x' \in \Sigma\}$
- Fixed lower boundary: $\Sigma_0 = \{x_n = 0\}$
- Time-dependent fluid domain: $\Omega_{b+\zeta(\cdot,t)} = \{0 < x_n < b + \zeta(x', t)\}$

Model single layer traveling waves problem

Main questions

1. Does traveling bulk force/surface stress induce traveling wave solutions to the free boundary incompressible Navier-Stokes equations?
2. If so, what is the natural container space for the traveling wave solution?

- Unknowns here are the fluid velocity, fluid pressure, and the upper free surface function.
- Non-trivial bulk force and/or surface stress are required to produce non-trivial solutions in Sobolev-type spaces

The free boundary incompressible Navier Stokes equations

$$
\begin{cases}
\rho(\partial_t w + w\cdot\nabla w) - \mu\Delta w + \nabla P = -\rho\mathfrak{g}e_n + \tilde{\mathfrak{f}}, & \text{in } \Omega_{b+\zeta(\cdot,t)}\\
\operatorname{div} w = 0, & \text{in } \Omega_{b+\zeta(\cdot,t)}\\
\partial_t\zeta = w\cdot\nu\sqrt{1+|\nabla'\zeta|^2}, & \text{on } \Sigma_{b+\zeta(\cdot,t)}\\
(PI - \mu\mathbb{D}w)\nu = [-\sigma\mathcal{H}(\zeta)I + P_{ext}I + \tilde{\mathcal{T}}]\nu, & \text{on } \Sigma_{b+\zeta(\cdot,t)}\\
w = 0, & \text{on } \Sigma_0.
\end{cases}
$$

- The first equation asserts the Newtonian balance of forces (conservation of momentum).
- The second enforces the conservation of mass.
- The third equation is the kinematic boundary condition describing the evolution of the free surface with the fluid.
- The fourth equation is the dynamic boundary condition asserting the balance of forces/stresses on the free surface.
- Last equation is the no-slip boundary condition.

The free boundary incompressible Navier Stokes equations

$$
\begin{cases}
\rho(\partial_t w + w \cdot \nabla w) - \mu \Delta w + \nabla P = -\rho \mathfrak{g} e_n + \tilde{\mathfrak{f}}, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\
\operatorname{div} w = 0, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\
\partial_t \zeta = w \cdot \nu \sqrt{1 + |\nabla' \zeta|^2}, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\
(PI - \mu \mathbb{D} w)\nu = [-\sigma \mathcal{H}(\zeta) I + P_{ext} I + \tilde{\mathcal{T}}]\nu, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\
w = 0, & \text{on } \Sigma_0.
\end{cases}
$$

- Constant gravitational force: $\mathbb{R}^n \ni \mathfrak{G} = -\rho \mathfrak{g} e_n$
- Constant external pressure $P_{ext} \in \mathbb{R}$
- Bulk force $\tilde{\mathfrak{f}}(\cdot, t) : \Omega_{b+\zeta(\cdot,t)} \to \mathbb{R}^n$
- Externally applied surface stress tensor $\tilde{\mathcal{T}} : \Sigma_{b+\zeta(\cdot,t)} \to \mathbb{R}^{n \times n}_{\mathrm{sym}}$
- Surface tension on free surface: $-\sigma H(\zeta)$, where

$$\mathcal{H}(\zeta) = \operatorname{div}' \left(\frac{\nabla' \zeta}{\sqrt{1 + |\nabla' \zeta|^2}} \right)$$

  is the mean-curvature of $\Sigma_{b+\zeta(\cdot,t)}$.

The free boundary incompressible Navier Stokes equations

$$\begin{cases} \partial_t w + w \cdot \nabla w - \Delta w + \nabla P = -e_n + \tilde{\mathfrak{f}}, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\ \operatorname{div} w = 0, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\ \partial_t \zeta = w \cdot \nu \sqrt{1 + |\nabla' \zeta|^2}, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\ (PI - \mathbb{D}w)\nu = [-\sigma \mathcal{H}(\zeta) I + P_{ext} I + \tilde{\mathcal{T}}]\nu, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\ w = 0, & \text{on } \Sigma_0. \end{cases}$$

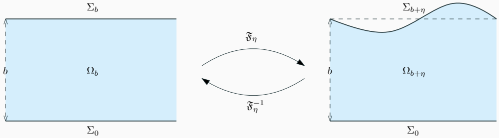
- Upon rescaling and renaming the parameters, we may assume $\rho = \mu = \mathfrak{g} = 1$.
- $\mathbb{D}w = (\nabla w) + (\nabla w)^T \in \mathbb{R}^{n\times n}_{\text{sym}}$ is the symmetrized gradient of w
- $\nu(\cdot, t) = (-\nabla' \zeta(\cdot, t), 1)/\sqrt{1 + |\nabla' \zeta(\cdot, t)|^2} \in \mathbb{R}^n$ is the outward pointing unit normal to the surface $\Sigma_{b+\zeta(\cdot,t)}$
- Note: both the system and the boundary depends on the unknown free surface ζ.

Traveling wave ansatz

- Assume that the bulk force and the external stress force are stationary in a moving coordinate system traveling at a constant velocity parallel to the flat rigid surface Σ_0: $\tilde{\mathfrak{f}}(x,t) = \mathfrak{f}(x - \gamma t e_1), \tilde{\mathcal{T}}(x,t) = \mathcal{T}(x' - \gamma t e_1)$.
- The speed of the traveling wave is $|\gamma|$, and $\operatorname{sgn}(\gamma)$ indicates the direction of travel along the e_1 axis.
- The stationary free surface is given by an unknown function $\eta : \mathbb{R}^n \to (-b, \infty)$, and it is related to ζ via $\zeta(x', t) = \eta(x' - \gamma t e_1)$.
- The stationary velocity, pressure v, q are related to w, P via

$$w(x,t) = v(x - \gamma t e_1), P(x,t) = q(x - \gamma t e_1).$$

Flattening



- Reformulate the problem on a fixed domain $\Omega_b = \Sigma \times (0, b)$, at the cost of worsening the nonlinearities in the system.
- Introduce the flattening map $\mathfrak{F}_\eta : \overline{\Omega_b} \to \overline{\Omega_{b+\eta}}$ associated to a continuous function $\eta : \Sigma \to (-b, \infty)$, defined via

$$\mathfrak{F}_\eta(x', x_n) = x + \frac{x_n \eta(x')}{b} e_n.$$

- Introduce the matrix $\mathcal{A} : \Omega_b \to \mathbb{R}^{n\times n}$ defined via

$$\mathcal{A}(x) = (\nabla \mathfrak{F}_\eta)^{-\intercal} = \begin{pmatrix} I_{(n-1)\times(n-1)} & \frac{-x_n \nabla' \eta(x')}{b+\eta(x')} \\ 0_{1\times(n-1)} & \frac{b}{b+\eta(x')} \end{pmatrix}.$$

- Use $\mathcal{A}$ to define the $\mathcal{A}$-differential operators $\nabla_{\mathcal{A}}, \operatorname{div}_{\mathcal{A}}, \Delta_{\mathcal{A}}$, etc. This reduces the original problem to a quasilinear problem in a fixed domain.

Previous work

Earliest work on traveling waves is for the inviscid problem (Euler equation)

- 2D irrotational: Nekrasov, Levi-Civita, Krasovskiĭ, Keady-Norbury, Toland, Amick-Toland, Amick-Fraenkel-Toland, Plotnikov-Toland, Beale
- 2D rotational: Constantin-Strauss, Wahlén, Walsh, Hur, Groves-Wahlén, Wheeler, Chen-Walsh-Wheeler
- 2D surface force: Wheeler, Walsh-Bühler-Shatah
- 3D irrotational: Iooss-Plotnikov, Groves-Sun, Buffoni-Groves-Sun-Wahlén

Previous work

Much less is known for the viscous problem.

- Stationary solutions: Jean, Pileckas, Gellrich, Nazarov-Pileckas, Pileckas-Zaleskis , Bae-Cho
- Non-stationary without free boundary: Chae-Dubovskiĭ, Freistühler, Kagei-Nishida, Escher-Lienstromberg, Zhuang-Escher
- Non-stationary with free boundary (experimental): Akylas-Cho-Diorio-Duncan, Masnadi-Duncan, Park-Cho

Viscous traveling waves

The model viscous traveling waves problem was studied by G. Leoni and I. Tice. To the best of our knowledge, their work produced the first rigorous construction of viscous traveling waves to the free boundary incompressible problem.

Below is a extremely rough and imprecise summary of their result.

Theorem (Leoni-Tice, CPAM '23)

Assuming positive surface tension in dimensions $n \geqslant 2$ and without surface tension in $n = 2$, for every non-zero traveling wave speed γ, there exists a nonempty open set of force and stress data that produces traveling wave solutions.

Remarks

The theorem only covers the case when $\gamma \neq 0$, i.e. the construction of non-stationary solutions.

Noah Stevenson (Princeton University) will talk about the stationary problem at 16:45.

Remarks

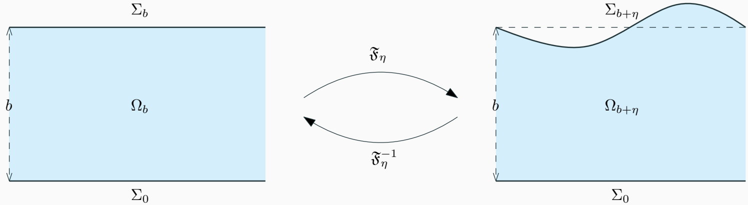
They show that the free surface function η belongs to a new scale of anisotropic Sobolev spaces denoted by X^s. There are a few natural mandates on these spaces by virtual of being the container space for the free surface function and the general proof strategy.

1. It behaves well with respect to composition with the flattening map.
2. It behaves well under the action of differential operators.
3. It obeys good product and multiplier estimates.
4. It enjoys good embedding properties into classical C_0^k spaces.

Understanding additional functional analytic properties of these spaces ends up being crucial in our analysis.

Rough summary of model approach

1. Reformulation: use a **traveling wave ansatz** and a **flattening map** to reformulate the problem on a fixed time-independent domain.



   The cost of such a reformulation is that the nonlinearities in the unflattened system are amplified and worsened.
2. Study an **overdetermined linearization** of the nonlinear problem around some solution. For the model problem, one can linearize around the equilibrium solution.

Rough summary of model approach

3. Identify **compatibility conditions** on the data by studying the formal adjoint of the overdetermined problem
4. Reformulate one of the compatibility conditions on the Fourier side in terms of special **symbols associated to specialized pseudodifferential operators**. This also gives a mechanism for the construction of the free surface function from data, and the container space for the free surface function is largely determined by the asymptotics of these special symbols.
5. Use the **implicit function theorem** coupled with the linear analysis to produce solutions to the nonlinear problem (important note: compatibility conditions from the linear analysis persists)

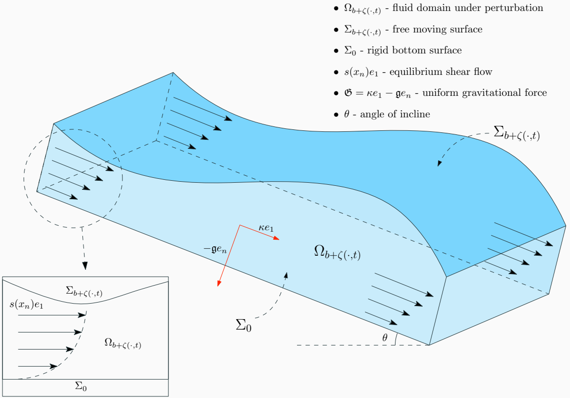
Recent work

Over the last few years multiple authors have studied variants of this model problem.

- N. Stevenson and I. Tice
  - Multilayer problem (SIMA '21)
  - Compressible problem (preprint '23)
  - Stationary and slowly traveling waves (preprint '23)
- H.Q. Nguyen and I. Tice
  - One-phase Muskat problem (ARMA '24)
- J. K. and I. Tice
  - Inclined problem with periodization (JFA '23)
  - Navier-slip problem (preprint '23)

The inclined problem with periodization

Fluid domain



- Choose e_n to be orthogonal to inclined surface
- Choose e_1 so gravitational field resolves into a horizontal and vertical component: $\mathbb{R}^n \ni \mathfrak{G} = \kappa e_1 - \mathfrak{g} e_n$, $\kappa \in \mathbb{R}$ and $\mathfrak{g} \in (0, \infty)$.
- The angle of incline $\theta \in (-\pi/2, \pi/2)$ is defined via $\tan\theta = \kappa/\mathfrak{g}$.

The inclined Navier-Stokes system

$$
\begin{cases}
\partial_t w + w \cdot \nabla w - \Delta w + \nabla P = -e_n + \kappa e_1 + \tilde{\mathfrak{f}}, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\
\operatorname{div} w = 0, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\
\partial_t \zeta = w \cdot \nu \sqrt{1 + |\nabla' \zeta|^2}, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\
(PI - \mathbb{D}w)\nu = [-\sigma \mathcal{H}(\zeta) I + P_{ext} I + \tilde{\mathcal{T}}]\nu, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\
w = 0, & \text{on } \Sigma_0.
\end{cases}
$$

- In the incline problem, the lineraization is around *shear flows*.
- We allow for a general cross-section Σ as a Cartesion product of $\mathbb{R}$-factors and $\mathbb{T}$-factors with different characteristic lengths to model periodization.
- The reduction steps introduce multiple new terms in the flattened system associated to the unflattened problem, most are harmless but some become a significant issue at the nonlinear level (compatibility condition).

Main result: inclined problem with periodization

Here we give a simplified version of the main result.

Theorem (K.-Tice '22)

Suppose that $\mathbb{N} \ni s > \frac{n}{2}$ and Σ is admissiable. Further suppose that either $\sigma > 0$ and $n \geqslant 2$ or else $\sigma = 0$ and $n = 2$. Then there exists open sets

$$\mathcal{U}^s \subset (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \times H^{s+2}(\Sigma \times \mathbb{R}; \mathbb{R}^{n\times n}_{\mathrm{sym}}) \times H^{s+1}(\Sigma \times \mathbb{R}; \mathbb{R}^n)$$

and an open set $\mathcal{O}^s$ belonging to a Sobolev-type space $\mathcal{X}^s$ such that the following hold.

1. $(0,0,0) \in \mathcal{O}^s$, *and for every* $(u,p,\eta) \in \mathcal{O}^s$ *we have that*
$$u \in C_b^{2+\left\lfloor s-\frac{n}{2}\right\rfloor}(\Omega;\mathbb{R}^n), p \in C_b^{1+\left\lfloor s-\frac{n}{2}\right\rfloor}(\Omega;\mathbb{R}), \eta \in C_0^{3+\left\lfloor s-\frac{n}{2}\right\rfloor}(\Sigma;\mathbb{R})$$
2. *For each* $(\gamma,\kappa,\mathcal{T},\mathfrak{f}) \in \mathcal{U}^s$, *there exists a unique* $(u,p,\eta) \in \mathcal{O}^s$ *classically solving the quasilinear* κ*-dependent flattened system.*
3. *The map* $\mathcal{U}^s \ni (\gamma,\kappa,\mathcal{T},\mathfrak{f}) \mapsto (u,p,\eta) \in \mathcal{O}^s$ *is* C^1 *and locally Lipschitz.*

Main difficulties and key techniques

1. Dependence on the incline parameter κ
   - Fix $\gamma \neq 0$ and perturb around $\kappa = 0$.
2. Periodization
   - Since we mandate elements of $X^s(\Sigma)$ to be classical functions, the space then needs to be at least be Banach. If it is not complete, taking its completion is insufficient as we recover equivalence classes of tempered distributions modulo polynomials. Understanding this for general Σ turns out to be quite subtle.
3. Compatibility condition for the nonlinear analysis
   - Compatibility conditions coming from the linear analysis mandates X^s to be a Banach algebra (at least for physically relevant cases $d = 2, 3$). We used Littlewood-Paley type techniques inspired by the work of Guo-Huang-Pausader-Widmayer on the rotational 3D Euler equations to prove this.

Semi-explicit estimate κ

- Define the set of (γ, κ) parameters for which we can produce solutions to be

  $$\mathfrak{P}^s = \{(\gamma, \kappa) \in \mathbb{R}\backslash\{0\} \times \mathbb{R} \mid (\gamma, \kappa, \mathcal{T}, \mathfrak{f}) \in \mathcal{U}^s \text{ for some } (\mathcal{T}, \mathfrak{f})\}.$$

  We show that for every $\gamma \in \mathbb{R}\backslash\{0\}$, there exists a $\kappa_0(\gamma) > 0$, depending on γ and the other physical parameters in a semi-explicit way, such that

  $$(-\kappa_0(\gamma), \kappa_0(\gamma)) \subseteq \{\kappa \in \mathbb{R} \mid (\gamma, \kappa) \in \mathfrak{P}^s\}.$$

- The estimate also suggests that for each $\kappa \in \mathbb{R}\backslash\{0\}$, the set $\{\gamma \in \mathbb{R}\backslash\{0\} \mid (\gamma, \kappa) \in \mathfrak{P}^s\}$ is bounded, and possibly empty for large $|\kappa|$. This is conjectural and we did not attempt to prove this due to the complicated dependence on various operator norms.

The anisotropic Sobolev space $X^s(\Sigma)$

- For $s \geqslant 0$ and general Γ that is a Cartesian product of $\mathbb{R}$ and $\mathbb{T}$ factors, we define

$$X^s(\Gamma;\mathbb{R}) = \{f \in \mathscr{S}'(\Gamma;\mathbb{C}) \mid f = \overline{f}, \hat{f} \in L^1_{\mathrm{loc}}(\hat{\Gamma};\mathbb{C}), \\ \|f\|_{X^s} = \|\mu\langle\cdot\rangle^{s-1}\hat{f}\|_{L^2} < \infty\}$$

  where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$ is the Japanese bracket and

$$\mu(\xi) = \begin{cases} \frac{|\xi_1|}{|\xi|} + |\xi| & \text{for } \xi \neq 0 \\ 1 & \text{for } \xi = 0. \end{cases}$$

- Since $1 \asymp \langle\cdot\rangle$ on $B(0,1)$, low frequency control is provided by $\mu(\cdot)$.
- $\mu(\cdot) \asymp \langle\cdot\rangle$ on $B(0,1)^c$, so high frequency control is provided by the standard Sobolev multiplier $\langle\cdot\rangle^{2s}$.
- Requiring $\mu(0) = 1$ is only relevant when $\Sigma = \mathbb{T}^d$, so that $\mu(0) = \langle 0\rangle$.
- We can similarly define $X^s(\Gamma;\mathbb{C})$.

Admissibility of Σ

- The functional analytic properties of the space $X^s(\Gamma)$ is subtle in the general setting. For example, we note that if the first factor of Γ is $\mathbb{T}$, then the multiplier degenerates at the low frequencies as $\xi_1 = 0$ for all $\xi \in B(0,1)$.

  We show that in some important cases, $X^s(\Gamma)$ fails to be complete.

Γ	$X^s(\Gamma)$ is
$\mathbb{R}$, $\mathbb{T}$	Complete and equal to $H^s(\Gamma)$
$\mathbb{R}^2, \mathbb{T}^2, \mathbb{R} \times \mathbb{T}$	Complete
$\mathbb{T} \times \mathbb{R}$	Incomplete

Table 1: $X^s(\Gamma)$ in the physical dimensions $n = 2, 3$

Admissibility of Σ

- We were able to classify exactly when $X^s(\Gamma)$ fails to be complete.

Γ	$X^s(\Gamma)$ is
$\mathbb{R}^{n-1}$,	Complete
$\mathbb{T}^{n-1}$, $\mathbb{R}\times\mathbb{T}^{n-2}$	Complete and equal to $H^s(\Gamma)$
$\mathbb{R}\times Y, Y \neq \mathbb{T}^{n-2}$	Complete and $H^s(\mathbb{R}\times Y) \subsetneq X^s(\mathbb{R}\times Y)$
$\mathbb{T}\times Y, Y \neq \mathbb{T}^{n-2}$	Incomplete iff. Y contains less than 3 $\mathbb{R}$-factors

Table 2: $X^s(\Gamma)$ in general dimensions $n \geqslant 2$

- $\Gamma = \mathbb{R}^{n-1}$ is studied in L-T '23
- If $\Gamma = \mathbb{T}^{n-1}$, the only low frequency mode is $\xi = 0$ and $\mu(0) = \langle 0 \rangle$, so $\mu(\xi) \asymp \langle \xi \rangle$ for all $\xi \in \hat{\Gamma}$.
- If $\Gamma = \mathbb{R}\times\mathbb{T}^{n-2}$, $\xi \in B(0,1) \iff \xi = (\xi_1, 0)$, and $\mu(\xi) = 1 + |\xi_1| \asymp \langle \xi \rangle$ on $B(0,1)$ and $\mu(\xi) \asymp \langle \xi \rangle$ for all $\xi \in \hat{\Gamma}$.
- If $\Gamma = \mathbb{T}\times\mathbb{R}$, $\xi \in B(0,1) \iff \xi = (0, \xi_2)$, and $\mu(\xi) = |\xi_2|$, so we have $\dot{H}^1$ type control and $\dot{H}^1(\mathbb{R}^d)$ is complete iff. $d > 2$.

The Navier-slip problem

The free boundary incompressible Navier-Stokes equations with Navier-slip conditions

$$\begin{cases} \rho(\partial_t w + w \cdot \nabla w) - \mu\Delta w + \nabla P = -\rho\mathfrak{g}e_n + \tilde{\mathfrak{f}}, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\ \operatorname{div} w = 0, & \text{in } \Omega_{b+\zeta(\cdot,t)} \\ \partial_t\zeta = w \cdot \nu\sqrt{1+|\nabla'\zeta|^2}, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\ (PI - \mu\mathbb{D}w)\nu = [-\sigma\mathcal{H}(\zeta)I + P_{ext}I + \tilde{\mathcal{T}}]\nu, & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\ \alpha[(PI - \mu\mathbb{D}w)\nu]' = w', & \text{on } \Sigma_0 \\ w_n = 0, & \text{on } \Sigma_0, \end{cases}$$

- $\alpha[(PI - \mu\mathbb{D}w)\nu]' = w'$ is referred to as the Navier-slip condition, asserting that the tangential fluid velocity is proportional to the tangential stress experienced by the fluid. $\alpha \geqslant 0$ is the characteristic slip length, $\alpha = 0$ corresponds to the no-slip problem.
- We can study the more general Navier-slip condition $\alpha[(PI - \mu\mathbb{D}w)\nu]' = [A(w)]'$ where A is smooth and satisfies certain coercivity conditions.

Main result: Navier-slip problem

Below is a simplified version of the main theorem.

Theorem (K.-Tice '23)

Suppose that $\mathbb{N} \ni s \geqslant 1 + \lfloor n/2 \rfloor$, $\alpha > 0$ *and that either* $\sigma > 0$ *and* $n \geqslant 2$ *or else* $\sigma = 0$ *and* $n = 2$. *Then there exists open sets*

$$\mathcal{U}^s \subset \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \times H^{s+3}(\mathbb{R}^n; \mathbb{R}^{n\times n}_{\mathrm{sym}}) \times H^{s+2}(\mathbb{R}^n; \mathbb{R}^n)$$

and $\mathcal{O}^s \subset \mathcal{X}^s_\alpha$ *such that for each* $\gamma_* \in \mathbb{R} \setminus \{0\}$, *there exists an open set* $V(\gamma_*)$ *such that for all* $\alpha_* \in (0,1)$ *the following hold.*

1. $(0,0,0) \in \mathcal{O}^s$, *and for every* $(u,p,\eta) \in \mathcal{O}^s$ *we have that*

$$u \in C_b^{2+\lfloor s - \frac{n}{2} \rfloor}(\Omega; \mathbb{R}^n), p \in C_b^{1+\lfloor s - \frac{n}{2} \rfloor}(\Omega; \mathbb{R}), \eta \in C_0^{3+\lfloor s - \frac{n}{2} \rfloor}(\mathbb{R}^{n-1}; \mathbb{R})$$

2. $(\alpha_*, \gamma_*, 0, 0) \in (0,1) \times V(\gamma_*) \subset \mathcal{U}^s$.

Recovering the no-slip condition

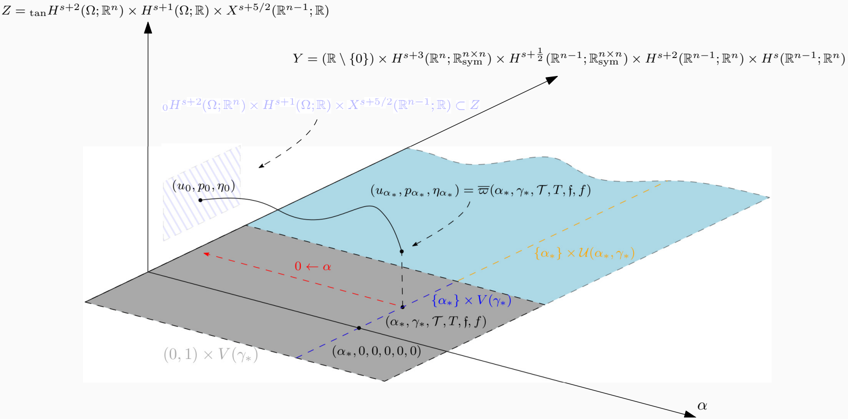
Theorem (K.-Tice '23)

3. *For every* $(\mathcal{T}, \mathfrak{f})$ *such that* $(\gamma_*, \mathcal{T}, \mathfrak{f}) \in V(\gamma_*)$*, there exists a unique solution tuple* $(u_{\alpha_*}, p_{\alpha_*}, \eta_{\alpha_*})$ *belonging to* $\mathcal{O}^s$ *classically solving the flattened system. Furthermore,* $(u_{\alpha_*}, p_{\alpha_*}, \eta_{\alpha_*}) \rightharpoonup (u_0, p_0, \eta_0)$ *weakly in* $H^{s+2}(\Omega; \mathbb{R}^n) \times H^{s+1}(\Omega; \mathbb{R}) \times X^{s+5/2}(\mathbb{R}^{n-1}; \mathbb{R})$ *as* $\alpha_* \to 0$*, where*

$$(u_0, p_0, \eta_0) \in C_b^{2+\left\lfloor s-\frac{n}{2}\right\rfloor}(\Omega; \mathbb{R}^n) \\ \times \in C_b^{1+\left\lfloor s-\frac{n}{2}\right\rfloor}(\Omega; \mathbb{R}) \times C_0^{3+\left\lfloor s-\frac{n}{2}\right\rfloor}(\mathbb{R}^{n-1}; \mathbb{R})$$

is the unique no-slip solution belonging to $\mathcal{O}^s$*.*

Picture sending $\alpha \to 0$



Key: in order to push $\alpha \to 0$ for all other data fixed, we need a uniform α-independent "core."

Main difficulties and key techniques

1. Dependence of the parameter α.
   - For a fixed $\alpha > 0$, developing a well-possedness theory is relatively straightforward.
   - A parameter dependent implicit function theorem requires showing that the solution norms of $(u_\alpha, p_\alpha, \eta_\alpha)$ can be controlled uniformly if $\alpha \in (0, 1)$.
   - This requires a careful analysis of the linear analysis and the dependence of the asymptotics of the specialized symbols on α.
   - We also needed a modified flattening map to handle the α-dependnent nonlinear terms appearing on the lower fixed boundary.

Conclusion and future work

Summary and future work

- The framework introduced by G. Leoni and I. Tice in their initial work on viscous traveling waves has been successfully utilized in many variants of model problem.
- For the inclined problem with periodization, the functional analytic properties of the anisotropic Sobolev space X^s play a key role in the analysis.
- For the Navier-slip problem, carefully tracking the dependence of α is required to prove the convergence result.

Possible future work

- $b \to \infty$? Here we don't have immediate control over the low frequencies.
- $b \to 0$?
- $\mu \to 0$?

Thank you for your attention!